.

Non-nequilibrium model on Apollonian networks

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We investigate the Majority-Vote Model with two states $(-1, +1)$ and a noise q on Apollonian networks. The main result found here is the presence of the phase transition as a function of the noise parameter q. We also studies de effect of redirecting a fraction p of the links of the network. By means of Monte Carlo simulations, we obtained the exponent ratio γ/ν, β/ν, and $1/\nu$ for several values of rewiring probability p. The critical noise was determined q_c and U^* also was calculated. The effective dimensionality of the system was observed to be independent on p, and the value $D_{eff} \approx 1.0$ is observed for these networks. Previous results on the Ising model in Apollonian Networks have reported no presence of a phase transition. Therefore, the results present here demonstrate that the Majority-Vote Model belongs to a different universality class as the equilibrium Ising Model on Apollonian Network.

I. INTRODUCTION

The Ising model [1, 2] is commonly used as a benchmark to test and improve new algorithms and methods for computer simulation in models of Statistical Mechanics. For instance, Monte Carlo methods such as Metropolis [3], Swendsen-Wang [4], Wang-Landau [5], Single histogram [6] and Broad histogram [7] have all been used to calculate the critical exponents of this model. The Ising model has also been employed to study social behavior and many of these models and others can be found out in [8].

G. Grinstein *et al.* [9] have argued that non-equilibrium stochastic spin systems on regular square lattices (SL) with up-down symmetry, fall into the same universality class of the equilibrium Ising model. The correspondence was observed for several models that do not obey detailed balance and on other regular lattices [10–15]. The majority-vote model with two states (MV2) is a non-equilibrium model proposed by M.J. Oliveira in 1992 which does not obey detailed balance. This model follows a stochastic dynamics with local rules and with up-down symmetry, and on a regular lattice shows a second-order phase transition with critical exponents β, γ, ν consistent [13, 16] with those of the equilibrium Ising model [1]. However, Lima *et al.* [17] have studied MV2 on Voronoi-Delaunay random lattices with periodic boundary conditions, and they obtained exponents different from those obtained on regular lattices, in disagreement with the conjecture suggested by G. Grinstein *et al.* [9].

Simulations on both *undirected* and *directed* scale-free networks [18, 19, 21–24, 26], random graphs [27, 28] and social networks [29–31], have attracted interest of researchers from various areas. These complex networks have been studied extensively by Lima *et al.* in the context of discrete models (MV2, Ising, and Potts model) [32–37]. Recently, the equilibrium Ising model was studied on a class of hierarchical scale-free networks, namely the Apollonian Networks (ANs) [40, 41], and it was shown that, on these networks, no phase transition is observed for these models.

In the present work, we study the MV2 model on normal and redirected ANs. By means of numerical simulations we found that MV2 model in ANs network displays a clear second-order phase transition. This demonstrates that, on these networks, MV2 and the Ising model do not fall in the same universality class, therefore contradicting Grinstein hypothesis [9]. The remainder of our paper, is organized as follows. In section 2, we present our model and some details about the Monte Carlo simulation as well as the calculations performed in the evolution of the physical quantities. In section 3, we do an analysis over the simulations performed and discuss the obtained results. Finally in section 4, we present our conclusions and final remarks.

II. MODEL AND SIMULATION

Our network is ANs type composed of $N = 3 + (3^n - 1)/2$ nodes, where n is the generation number [38]. To introduce a level of disorder we redirect a faction p of the of the links. The redirecting results in a directed network, preserving the out-going node of the redirected link but changing the incoming node. In the limit of $p = 0$ we have the Apollonian Networks, while in the limit $p = 1$ we have something similar to random networks [27]. Note however, that the number of outgoing links of each node is

preserved, therefore, even in the limit $p = 1$ the network still have hubs that that are the most influent nodes. For $p = 0$ we have the standard Apollonian Networks. These networks display a scale free degree distribution and a hierarchical structure. The critical properties of percolation and Potts models on these networks have been investigated [41, 42] and its was shown that in the thermodynamic limit there is no phase transition, with the ordered phase prevailing for any finite temperature.

On the MV2 model, the system dynamics is as follows. Initially, we assign a spin variable σ with values ± 1 at each node of the network. At each step we try to spin flip a node. The flip is accepted with probability

$$w_i(\sigma) = \frac{1}{2}\left[1 - (1 - 2q)\sigma_i S\left(\sum_{\delta=1}^{k_i} \sigma_{i+\delta}\right)\right], \qquad (1)$$

where $S(x)$ is the sign ± 1 of x if $x \neq 0$, $S(x) = 0$ if $x = 0$. To calculate w_i our sum runs over the number k of nearest neighbors of i-th spin. Eq. (1) means that with probability $(1 - q)$ the spin will adopt the same state as the majority of its neighbors. Here, the control parameter $0 \leq q \leq 1$ plays a role similar to the temperature in equilibrium systems, the smaller q greater the probability of parallel aligning with the local majority.

We performed Monte Carlo simulation on the ANs with various systems of size $N = 3,283;\ 9,844;\ 29,527;\ 88,576,$ and $265,723$. We wait 1×10^5 Monte Carlo steps (MCS) in order to reach the steady state, and then the time averages are estimated over the next 2×10^5 MCS. One MCS is accomplished after all the N spins are investigated whether they flip or not. We carried out $N_{sample} = 1,000$ to $10,000$ independent simulation runs for each lattice and for a given set of parameters (q, N).

To study the critical behavior of the model we define the variable $m = \sum_{i=1}^{N} \sigma_i/N$. In particular, we are interested in the magnetization M, susceptibility χ and the reduced fourth-order cumulant U_4

$$M(q) = \left[\langle|m|\rangle\right]_{av}, \qquad (2)$$

$$\chi(q) = N\left[\left(\langle m^2\rangle - \langle m\rangle^2\right)\right]_{av}, \qquad (3)$$

$$U_4(q) = 1 - \left[\langle m^4\rangle/\left(3\langle m^2\rangle^2\right)\right]_{av}, \qquad (4)$$

where $\langle \cdots \rangle$ stands for a thermodynamics average. The results are averaged over the 50 (av) ANs independent simulations. These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$M = N^{-\beta/\nu} f_m(x), \qquad (5)$$

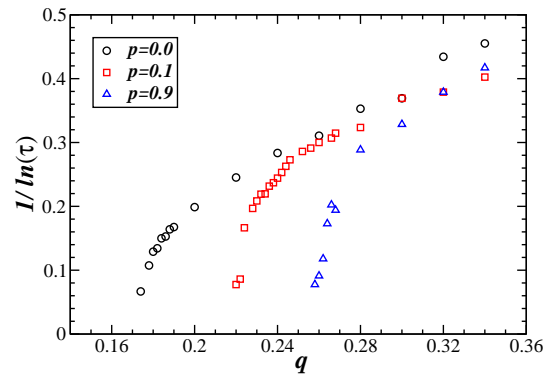$$\chi = N^{\gamma/\nu} f_\chi(x), \qquad (6)$$

FIG. 1. Reciprocal logarithm of the relaxation times τ on directed Apollonian Networks versus q for different redirecting probability ($\bigcirc$) $p = 0.0$, ($\square$) $p = 0.1$, ($\triangle$) $p = 0.9$.

$$\frac{dU}{dq} = N^{1/\nu} f_U(x), \qquad (7)$$

where ν, β, and γ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x = (q - q_c)N^{1/\nu} \qquad (8)$$

being the scaling variable. From this scaling relations we obtained the exponents β/ν and γ/ν, respectively. Moreover, the value of q^* for which χ has a maximum is expected to scale with the system size as

$$q^* = q_c + bN^{-1/\nu} \qquad (9)$$

where $b \approx 1$. Therefore, these relations may be used to obtain the exponent $1/\nu$.

III. RESULTS AND DISCUSSION

In order to test, if there is a phase transition in MV2 models we measured the relaxation time τ as a funcion of the noise parameter q. We start the system with all spins up, a number of spins equal to $7,174,456$ ($G15$). We determine the time τ after which the magnetization has flipped its sign for the first time, and then take the median value of nine samples. As one can see in Fig. 1, the relaxation time goes to infinity at some positive q value, indicating a second order phase transition. On contrast, the Ising model on directed Barabasi-Albert [21] networks has no phase transition and agrees with the modified Arrhenius law for relaxation time [25].

In Fig. 2, we show the dependence of the magnetization M, Binder's cumulant U_4 and susceptibility χ on the noise parameter q, obtained from simulations on ANs with $N = 3,283;\ 9,844;\ 29,527;\ 88,576,$ and $265,723$ sites and with $n = 8,\ 9,\ 10,\ 11,$ and 12 generation ($G8$, $G9$, $G10$, $G11$, and $G12$). The shapes of magnetization curves atest the presence of a second-order phase transition in the system. The critical noise parameter q_c is
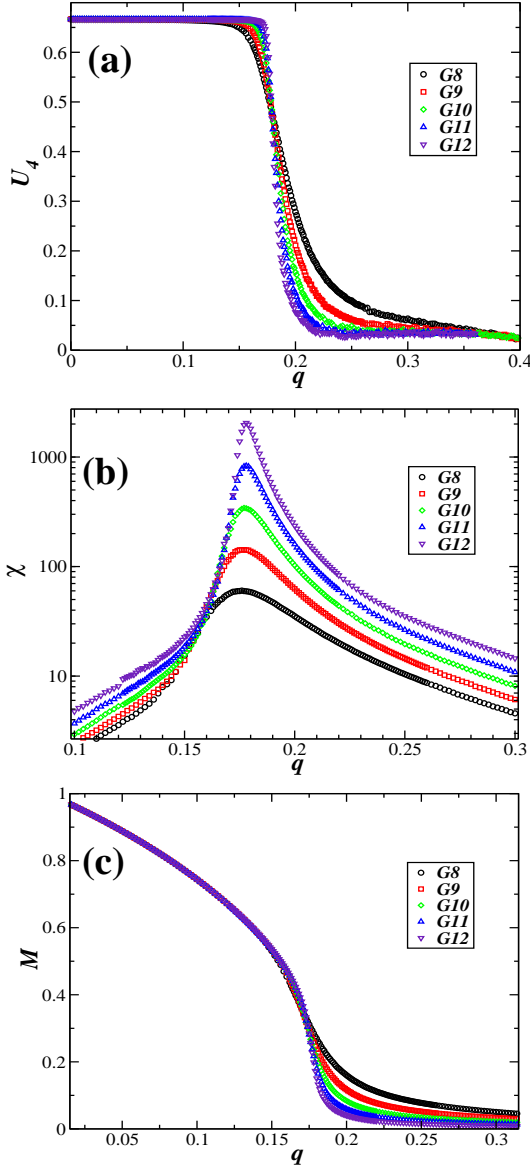
FIG. 2. We show the state functions U_4 (a) , χ (b), and M (c) studied here as a function of the noise q for the Apolloanian Network (redirecting probability $p = 0$). The results are presented for different generations of the Appolonian Network, ($\bigcirc$) $G8$, ($\square$) $G9$, ($\diamondsuit$) $G10$, ($\triangle$) $G11$, ($\triangledown$) $G12$. The Binder cumulant clearly presents a the phase-transition, with the critical noise being obtained by the point where the curves intercept each other, $q_c \approx 0.17$. The signature of the phase transition is also observed in the curves for the susceptibility and magnetization.

estimated as the point where the curves for different system sizes N intercept each other [43]. The obtained values for q_c can be seen in table 1. The critical exponents β/ν and γ/ν can be obtained by investigating the scaling at criticality of the magnetization and susceptibility, respectively. As shown in Fig. 3, both quantities scale as power laws with the controlling exponents depending on the redirecting probability. We use Eq. (9) to obtain the
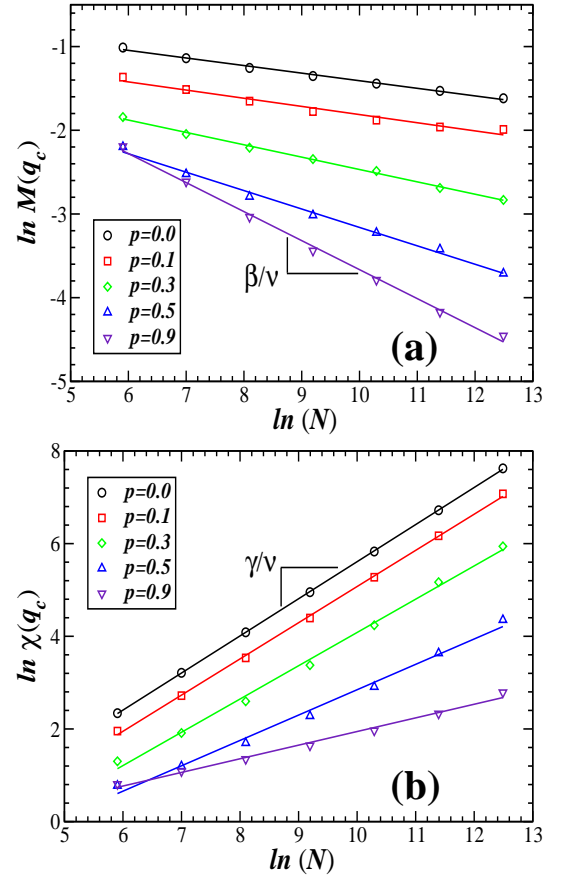

FIG. 3. The Log-log plot of the magnetization $M(q_c)$ (a), susceptibility $\chi(q_c)$ (b) as a function of the number of nodes N on the Apollonian Networks. Each curve of magnetization and susceptibility were calculated at the critical point q_c for each value of p as shown in table 1. The different symbols correspond to following redirecting probabilities: ($\bigcirc$) $p = 0.0$, ($\square$) $p = 0.1$, ($\diamondsuit$) $p = 0.3$, ($\triangle$) $p = 0.5$ and ($\triangledown$) $p = 0.9$. In both graphs (a) and (b) the solid lines represent the best linear fit for each value of p and the slopes give β/ν and γ/ν, respectively.

exponent $1/\nu$ as shown in Fig. 4. In table 1 we summarize all critical exponents for each values of p. In Fig. 5 we plot respectively the susceptibility and magnetization as a function of q for systems with redirecting probability $p = 0.1$ and different generations of the Apollonian Networks. Using the scaling exponents and Eqs. (2) and (3) we produced the collapsed data shown in the insets, which confirm the accuracy of the exponents.

We can also compute the effective dimensionalities of the system, defined as $D_{eff} = 2\beta/\nu + \gamma/\nu$ [44]. For all values of p we obtain $D_{eff} \approx 1$, as seen in table 1. This behavior has been previously observed for MV2 and MV3 on various Scale-free networks and on Erdös-Rényi random graphs [27].
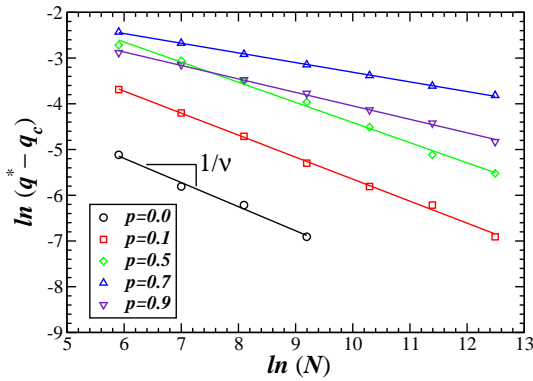
FIG. 4. Log-log plot of the displacement of the point of maximum of the susceptibility ($q^* - q_c$) agains the number of nodes N on the Apollonian Networks. The symbols represent the following values of p: ($\bigcirc$) $p = 0.0$, ($\square$) $p = 0.1$, ($\Diamond$) $p = 0.5$, ($\triangle$) $p = 0.7$ and (∇) $p = 0.9$. The solid lines indicate the best linear regression and from the straigh lines slopes we obtain the exponent $1/\nu$ according to Eq. (9).

p	q_c	$1/\nu$	β/ν	γ/ν	Deff
0.0	0.178(3)	0.53(4)	0.091(3)	0.80(2)	0.98(3)
0.1	0.223(5)	0.48(2)	0.097(5)	0.79(3)	0.98(3)
0.2	0.249(3)	0.66(3),	0.112(3)	0.80(3)	1.02(2)
0.3	0.271(5)	0.61(8)	0.148(5)	0.72(5)	1.02(3)
0.4	0.284(5)	0.71(7)	0.130(3)	0.69(4)	0.95(6)
0.5	0.296(4)	0.44(5)	0.220(8)	0.55(3)	0.99(5)
0.6	0.313(3)	0.23(3)	0.343(4)	0.32(2)	1.01(3)
0.7	0.311(5)	0.21(5)	0.374(5)	0.25(3)	1.01(4)
0.8	0.290(5)	0.27(3)	0.36(2)	0.28(3)	1.00(2)
0.9	0.2629(3)	0.29(6)	0.347(9)	0.29(2)	0.98(5)

TABLE I. The critical noise q_c, and the critical exponents, for ANs with probability p. Error bars are statistical only.

IV. CONCLUSION

We presented results for the non-equilibrium MV2 on ANs. On this network, the non-equilibrium MV2 shows a well defined second-order phase transition. On other hand, it has been previously shown that the equilibrium Ising model does not present a phase transition in these networks [40, 41]. Therefore our results demonstrate that MV2 model on ANs belongs to another universality class, in disagreement with the conjecture of Grinstein et al. [9]. The source of this distinction is due to the different behavior of noise in each of these models. In the Ising model, the probability of switching a highly connected spin against the local majority is smaller than a less connected one; since the energy variation is larger for a more connected spin. In the MV2 model, the probability of a spin switching against the local majority is always given by q, independent on the number of neighbors of this spin. Interestingly, the effective dimensionality of
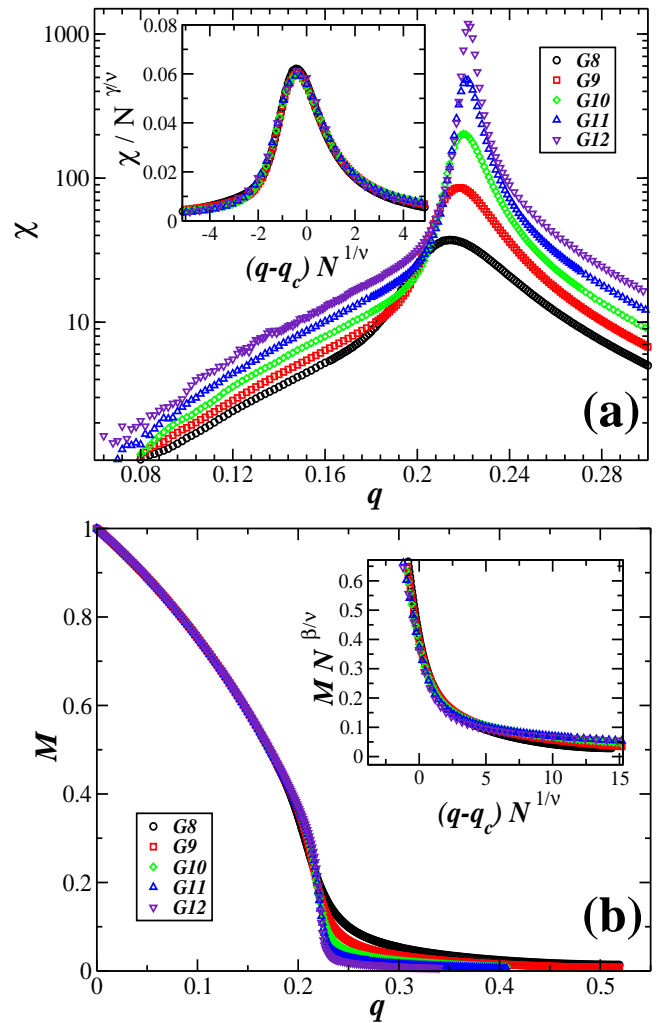

FIG. 5. We show the state functions χ (a) and M (b) as a function of the noise q for directed Apolloanian Networks (redirecting probability $p = 0.1$). The results are presented for different generations of the Appolonian Network,($\bigcirc$) $G8$, ($\square$) $G9$, ($\Diamond$) $G10$, ($\triangle$) $G11$, (∇) $G12$. In the insets we show the rescaled data, collapsed using the critical exponents obtained from table 1.

the system, defined as $D_{eff} = 2\beta/\nu + \gamma/\nu$, is always a value close 1.0 independent of the rewiring probability p, as seen in table 1. This value for D_{eff} has already been previously obtained for MV2 and MV3 on various Scale-free networks and on Erdös-Rényi random graphs.

A. Acknowledgments

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